Duplication Formula:

Prove that

$$\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n),$$

where n is an integer.

Proof Before proving the Duplication Formula we prove that

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$$

Since $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Let $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$, then

we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-1}\theta \cdot \sin\theta \cdot \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

or $$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

or $$\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \quad \text{--- (1)}$$

putting $m = \frac{1}{2}$ in (1), we get

$$\int_0^{\pi/2} \cos^{2n-1}\theta \, d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{2\,\Gamma\left(\frac{1}{2}+n\right)}$$

$$= \frac{\sqrt{\pi}\cdot\Gamma(n)}{2\,\Gamma\left(\frac{2n+1}{2}\right)} \qquad \left[\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}\;\Gamma(n)}{2\cdot\Gamma\left(\frac{2n+1}{2}\right)} \quad \text{--- (2)}$$

Again, putting $m = n$ in (1)

$$\int_0^{\pi/2} \sin^{2n-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{\{\Gamma(n)\}^2}{2\,\Gamma(2n)} \quad \text{--- (3)}$$

$$\Rightarrow \frac{\{\Gamma(n)\}^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}} \int_0^{\pi/2} (2 \sin\theta . \cos\theta)^{2n-1} d\theta$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} (\sin 2\theta)^{2n-1} d\theta \quad \text{--- (4)}$$

Let $z = 2\theta$ so that

$dz = 2d\theta$

$\therefore$ R.H.S of (4)

$$= \frac{1}{2^{2n-1}} \int_0^{\pi} (\sin z)^{2n-1} \frac{dz}{2}$$

$$= \frac{1}{2^{2n}} \int_0^{\pi} \sin^{2n-1} z \, dz$$

$$= \frac{1}{2^{2n}} \times 2 \int_0^{\pi/2} \sin^{2n-1} z \, dz$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} z \, dz$$

$$= \frac{\{\Gamma(n)\}^2}{2\Gamma(2n)} \quad \text{From (4)}$$

$$\therefore \frac{\{\Gamma(n)\}^2}{2\Gamma(2n)} = \frac{1}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1} z \, dz \quad \text{--- (5)}$$

$$\therefore \frac{\{\Gamma(n)\}^2}{2\,\Gamma(2n)} = \frac{1}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1}\left(\frac{\pi}{2}-z\right) dz$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1}\left(\frac{\pi}{2}-z\right) dz \qquad \left\{ \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right\}$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi/2} \cos^{2n-1} z \, dz$$

$$= \frac{1}{2^{2n-1}} \times \frac{\sqrt{\pi}\;\Gamma(n)}{2\cdot\Gamma\left(\frac{2n+1}{2}\right)} \qquad \{\text{From } (2)\}$$

$$\Rightarrow \frac{\{\Gamma(n)\}^2}{\cancel{2}\,\Gamma(2n)} = \frac{1}{2^{2n-1}} \times \frac{\sqrt{\pi}\;\Gamma(\cancel{n})}{\cancel{2}\cdot\Gamma\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \boxed{\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\;\Gamma(2n)}{2^{2n-1}}}$$

Above result is Known as Duplication Formula.